

Leading log expansion of system of combinatorial Dyson Schwinger equations

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Abstract

We study combinatorial Dyson Schwinger equations, expressed in the Hopf algebra of words with a quasi shuffle product. We map them into an algebra of polynomials in one indeterminate L and show that the leading log expansion one obtains with such a mapping are simple power law like expression.

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1 Introduction

Dyson Schwinger equations are functional equations in Quantum Field Theory fulfilled by the Green's functions, which are the expectation values of fields monomials and lead to some scattering amplitudes in particle physics. These equations are fix-point and self coherent equations. They also have analogies in statistical field theories. Dirk Kreimer, in his works about Hopf algebraic renormalization, extended these Dyson Schwinger equations into a combinatorial form, using a Hochschild-1-cohomology, often called B_+ operator, which plays the role of the linear forms in the functional expression. The equations become equations for series X^r 's expressed in a specified Hopf algebra. It leads to rich purely mathematical problems, which were for example studied by Loic Foissy in [6] or more recently by Joachim Kock in [5].

Here, we study a specific system of combinatorial Dyson Schwinger equations in the Hopf algebra of words endowed with a quasi shuffle product, in order to find at the end the leading series of the corresponding Green's functions, called leading log expansion. The Green's functions are supposed to depend of only one kinematical variable L , and one coupling constant α . This simplifies the correspondance, given by the so-called Feynman rules, between combinatorics

and physics. Given these settings, we write our CDSE's as:

$$X^r = \mathbf{1}_W + \text{sign}(\eta_r) \alpha B_+^{a_r}(X^r Q),$$

$$Q = \prod_{r'=1}^R (X^{r'})^{\eta_{r'}}.$$

The work done here is largely inspired by [7]. The reason of why we use this Hopf algebra of words is because it allows a nice factorization of words in components only linear in L once mapped by the Feynman rules.

We first start by giving the definitions and the elementary properties we will need. They are reminders on the theory of Hopf algebras and Hopf algebraic renormalization, developed in [3], [4], [8] and [9]. We then define what is the Hopf algebra of words that we will use to define our CDSE's, and finally pick out their leading log expansion G_{LL}^r thanks to a theorem proved in the Appendix. We finally show that:

$$\begin{aligned} G_{LL}^r &= (1 + A\alpha L)^{-\frac{c_r}{A}} \text{ if } \eta_r < 0, \\ G_{LL}^r &= (1 - A\alpha L)^{\frac{c_r}{A}} \text{ if } \eta_r > 0, \\ A &= \sum_{r'=1}^R \eta_{r'} c_{r'}. \end{aligned}$$

2 Preliminaries

We recall there some useful definitions and properties about Hopf algebras. If the reader is not familiar with the concept of Hopf algebra, [8] may help him a lot. It is also in this paper that one can find the proofs of the next properties.

Definition 1 A bialgebra $(H, m, \mathbf{1}, \Delta, \hat{\mathbf{1}})$ is a vector space H over a field \mathbb{K} together with an associative product $m : H \otimes H \rightarrow H$, a unit $\mathbf{1}$ such that $\forall x \in H, m(x \otimes \mathbf{1}) = m(\mathbf{1} \otimes x) = x$, a coassociative coproduct $\Delta : H \rightarrow H \otimes H$ and a counit $\hat{\mathbf{1}}$ satisfying $(Id \otimes \hat{\mathbf{1}}) \circ \Delta(x) = (\hat{\mathbf{1}} \otimes Id) \circ \Delta(x) = x$. Furthermore, Δ and $\hat{\mathbf{1}}$ have to be algebra morphisms with respect to the product m or equivalently, m and $\mathbf{1}$ have to be coalgebra morphisms with respect to the coproduct Δ .

Definition 2 If $(H, m, \mathbf{1}, \Delta, \hat{\mathbf{1}}, S)$ is a bialgebra together with a linear map $S : H \rightarrow H$ which fulfills $m(S \otimes Id) \circ \Delta = m(Id \otimes S) \circ \Delta = \mathbf{1}\hat{\mathbf{1}}$, it is called a Hopf algebra. S is called the antipode.

Definition 3 A bialgebra (respectively, a Hopf algebra) is called filtered iff there exist subspaces $H^0 \subset H^1 \subset \dots \subset H^n \subset \dots$ such that $\bigcup_n H^n = H$, $m(H^p \otimes H^q) \subset H^{p+q}$, $\Delta(H^n) \subset \sum_{p+q=n} H^p \otimes H^q$ (respectively, if furthermore $S(H^n) \subset H^n$).

Definition 4 A bialgebra (respectively, a Hopf algebra) is called graded iff there exist subspaces $H_0, H_1, \dots, H_n, \dots$ such that $\bigoplus_n H_n = H$, $m(H_p \otimes H_q) \subset H_{p+q}$, $\Delta(H_n) \subset \bigoplus_{p+q=n} H_p \otimes H_q$ (respectively, if furthermore $S(H_n) \subset H_n$).

Definition 5 A filtered (respectively graded) bialgebra or Hopf algebra is called connected iff H^0 (respectively H_0) is one-dimensional.

Remark: A graded bialgebra (respectively Hopf algebra) is in particular a filtered bialgebra (respectively Hopf algebra). The canonical filtration associated with a grading is given by, keeping the same notation as above:

$$H^n = \bigoplus_{j=0}^n H_j. \quad (1)$$

Definition 6 A bialgebra or a Hopf algebra is called pointed iff all its simple left (or right) comodules are one-dimensional.

Definition 7 Let $(A, m_A, \mathbf{1}_A)$ be a unital algebra and H be a bialgebra or a Hopf algebra as above. One defines the convolution product \star on $\text{Hom}(H, A)$ by:

$$\forall f_1, f_2 \in \text{Hom}(H, A), f_1 \star f_2 = m_A(f_1 \otimes f_2) \circ \Delta. \quad (2)$$

Remark: The convolution product admit a neutral element: $\mathbf{1}_A \hat{\mathbf{1}}$.

Remark: The antipode of a Hopf algebra is the inverse of the identity for the convolution product in $\text{Hom}(H, H)$.

Definition 8 An element $x \in H$ such that $\Delta(x) = \mathbf{1} \otimes x + x \otimes \mathbf{1}$ is called a primitive element. The subspace of H of primitive elements is denoted by $\text{Prim}(H)$.

Definition 9 The reduced coproduct $\tilde{\Delta}$ is defined by $\tilde{\Delta}(\mathbf{1}) = 0$ and by:

$$\forall x \neq \mathbf{1}, \tilde{\Delta}(x) = \Delta(x) - \mathbf{1} \otimes x - x \otimes \mathbf{1},$$

and the k^{th} power of the reduced coproduct by:

$$\tilde{\Delta}^k = \underbrace{(Id \otimes Id \otimes \dots \otimes Id \otimes \tilde{\Delta})}_{k-1 \text{ times}} \dots (Id \otimes \tilde{\Delta}) \tilde{\Delta}.$$

One can easily check that the reduced coproduct is coassociative.

Proposition 1 If H is a pointed bialgebra or a pointed Hopf algebra, one can endowed it with the coradical filtration:

$$x \in H^k \text{ iff } \tilde{\Delta}^k(x) = 0.$$

An element x such that $\tilde{\Delta}^k(x) = 0$ and $\tilde{\Delta}^{k-1} \neq 0$ is said to be of coradical degree k .

Proposition 2 If H is a connected filtered bialgebra, then it extends canonically to a Hopf algebra. The antipode is defined by $S = \sum_k (\mathbf{1}\hat{\mathbf{1}} - Id)^{\star k}$. It is given by $S(\mathbf{1}) = \mathbf{1}$ and for $x \neq \mathbf{1}$ recursively by one of the two following formulas:

$$S(x) = -x - (S \otimes Id)\tilde{\Delta}(x), \quad (3)$$

$$S(x) = -x - (Id \otimes S)\tilde{\Delta}(x). \quad (4)$$

2.1 Hochschild Cohomology

The work we present here can also be found in [9] or in [4].

One defines a Hochschild cochain complex:

$$\{Hom(H, H^{\otimes n})\}_{n \in \mathbb{N}} \quad (5)$$

with coboundary maps b_n such that if we let $L_n \in Hom(H, H^{\otimes n})$ and $h \in H$ then:

$$b_n L_n(h) = (id \otimes L_n) \Delta(h) + \sum_{i=1}^n (-1)^i \Delta_i L_n(h) + (-1)^{n+1} L(h) \otimes \mathbf{1}, \quad (6)$$

$$\Delta_i = \underbrace{(id \otimes \dots \otimes id \otimes \Delta \otimes id \otimes \dots \otimes id)}_{n \text{ times}} \text{ with } \Delta \text{ at the } i^{th} \text{-place.} \quad (7)$$

The elements of $\ker(b_{n+1})$ are called Hochschild-n-cocycles and the set of all such Hochschild-n-cocycles is denoted by ZH^n . The elements of $Im(b_n)$ are called Hochschild-n-coboundaries and the set of all such Hochschild-n-coboundaries is denoted by BH^n . Finally, the n^{th} cohomology space is $HH^n = \frac{ZH^n}{BH^n}$.

In this paper, we are interested by Hochschild-1-cocycles, which we will denote by B_+^x such that $B_+^x(\mathbf{1}) = x \in Prim(H)$. As we have for any $L_0 \in Hom(H, \mathbb{K})$

$$b_0 \circ L_0(\mathbf{1}) = L_0(\mathbf{1})\mathbf{1} - (id \otimes L_0) \circ \Delta(\mathbf{1}) = 0 \quad (8)$$

our B_+^x is in fact a Hochschild-1-cocycle. We also have

$$\Delta \circ B_+(\mathbf{1}) = B_+(\mathbf{1}) \otimes \mathbf{1} + \mathbf{1} \otimes B_+(\mathbf{1}), \quad (9)$$

i.e. $B_+(\mathbf{1}) \in Prim(H)$ for all Hochschild-1-cocycles B_+ .

2.2 Group of characters

Let $(A, m_A, \mathbf{1}_A)$ be a unital commutative algebra, $(H, m, \mathbf{1}, \Delta, \hat{\mathbf{1}}, S)$ be a Hopf algebra, and \star be the convolution product as before. Consider the set

$$G(H, A) = \{\phi \in Hom(H, A) / \phi(\mathbf{1}) = \mathbf{1}_A\}.$$

Elements of $G(H, A)$ are called characters.

Proposition 3 *The set $(G(H, A), \star, \mathbf{1}_A \hat{\mathbf{1}})$ is a group*

Furthermore, consider the set

$$g(H, A) = \{\sigma \in Hom(H, A) / \sigma(xy) = \sigma(x) \hat{\mathbf{1}}(y) + \hat{\mathbf{1}}(x) \sigma(y)\}$$

and the bracket $[\cdot, \cdot]$ with

$$\forall \sigma, \rho \in Hom(H, A), [\sigma, \rho] = \sigma \star \rho - \rho \star \sigma.$$

Proposition 4 *The set $(g(H, A, [,])$ is a Lie Algebra.*

Elements of $g(H, A)$ are called infinitesimal characters.

Remark: $\forall \sigma \in g(H, A), \sigma(\mathbf{1}) = 0$

Proposition 5

$$\forall \sigma \in g(H, A), \exp_\star(\sigma) = \sum_n \frac{\sigma^{\star n}}{n!} \in G(H, A). \quad (10)$$

$$\forall \phi \in G(H, A), \exists \sigma \in g(H, A)/\phi = \exp_\star(\sigma). \quad (11)$$

The sum in (10) is bounded by the coradical degree of $x \in H$ as $\sigma(\mathbf{1}) = 0$. The proofs of all these propositions can be found in [4] and in [8].

3 Hopf algebra of words

Let Ω be a countable space. We define a symmetric function: $\Omega \otimes \Omega \rightarrow \Omega$. We call inherited elements those which are in $Im(\Theta) \subset \Omega$. We define an equivalence relation:

$$E : \Theta(a, \Theta(b, c)) \sim \Theta(\Theta(a, b), c).$$

We let H_Θ be the quotient space Ω/E in which Θ is fully symmetric and in this space we write $\Theta(\Theta(a, b), c) \equiv \Theta(a, b, c)$. Finally, we suppose that there are $R \in \mathbb{N}^*$ fixed non inherited elements in H_Θ . The set of these non inherited elements is denoted by H_L .

We call the set H_Θ an alphabet and any of its element a letter. We call a word a concatenation of several letters and we denote by H_W the vector space generated by all words; each word defining a basis element. We say that a word is of length $l(w) \in \mathbb{N}$ if it's written as the concatenation of $l(w)$ letters. The length of a sum of words is defined to be the length of the word with the biggest length in the sum. We let \emptyset , the empty word, being the unique word of length 0. One define a grading on H_W , denoted by $|w|$, for any word w by:

$$|\emptyset| = 0, \quad (12)$$

$$\forall a \in H_L, \quad |a| = 1, \quad (13)$$

$$\forall a, b \in H_\Theta, \quad |\Theta(a, b)| = |a| + |b|, \quad (14)$$

$$\forall u, v \in H_W, \quad |uv| = |u| + |v|. \quad (15)$$

We define recursively a map \sqcup , called the shuffle product, by:

$$\sqcup : H_W \otimes H_W \rightarrow H_W \quad (16)$$

$$\forall w \in H_W, w \sqcup \emptyset = w, \quad (17)$$

$$\forall a_i u, a_j v \in H_W, a_i u \sqcup a_j v = a_i(u \sqcup a_j v) + a_j(a_i u \sqcup v). \quad (18)$$

Similarly, we define the quasi shuffle product, denoted by \sqcup_{Θ} ,

$$\sqcup_{\Theta} : H_W \otimes H_W \rightarrow H_W \quad (19)$$

$$\forall w \in H_W, w \sqcup_{\Theta} \emptyset = w, \quad (20)$$

$$\begin{aligned} \forall a_i u, a_j v \in H_W, a_i u \sqcup_{\Theta} a_j v &= a_i(u \sqcup_{\Theta} a_j v) + a_j(a_i u \sqcup_{\Theta} v) \\ &\quad + \Theta(a_i, a_j) u \sqcup_{\Theta} v. \end{aligned} \quad (21)$$

We let

$$\begin{aligned} \Delta : H_W &\rightarrow H_W \otimes H_W \\ w &\mapsto \sum_{uv=w} v \otimes u \end{aligned} \quad (22)$$

be the deconcatenation. Finally, we denote by δ_{\emptyset} the indicatrix of \emptyset .

We give an order on H_L by $a_i < a_j \Leftrightarrow i < j$ for $i, j \in \{1, \dots, R\}$. The order is extended on H_{Θ} by:

$$\forall a, b, c \in H_R, \quad a < \Theta(b, c), \quad (23)$$

$$\forall a, b, c \in H_{\Theta}, \quad \Theta(a, b) < \Theta(a, c) \Leftrightarrow b < c, \quad (24)$$

$$\forall a, b, c, d \in H_{\Theta}, \quad \Theta(a, b) < \Theta(c, d) \Leftrightarrow a < c. \quad (25)$$

We then extend this order to the words by saying $u < uv$ and $uav < ubv'$ with u, v, v' being words and a, b letters such that $a < b$. This order is often call the lexicographic order.

Definition 10 A word $w \in H_W$ is called *Lyndon* iff $\forall u, v \in H_W, w = uv \Rightarrow w < v$.

Proposition 6 $(H_W, \sqcup, \emptyset, \Delta, \delta_{\emptyset})$ and $(H_W, \sqcup_{\Theta}, \emptyset, \Delta, \delta_{\emptyset})$ are bialgebras, graded connected by $|\cdot|$. As algebras, They are freely generated by the Lyndon words.

This is proved in [1], [2].

Corollary 1 $(H_W, \sqcup, \emptyset, \Delta, \delta_{\emptyset}, S)$ and $(H_W, \sqcup, \emptyset, \Delta, \delta_{\emptyset}, S_{\Theta})$ are Hopf algebras, with S and S_{Θ} defined as in Proposition (2).

Remark: By definition, the shuffle product and the quasi shuffle product are commutative. They conserve the graduation $|\cdot|$. Furthermore, the length of words defines another graduation for $(H_W, \sqcup, \emptyset, \Delta, \delta_{\emptyset})$, and only a filtration for $(H_W, \sqcup, \emptyset, \Delta, \delta_{\emptyset}, S_{\Theta})$.

We denote by $C(n)$ a partition of $\{1, \dots, n\}, n \in \mathbb{N}$. We define an action of $C(n)$ over a word $w = a_{i_1} \dots a_{i_n}$ of length n by:

$$< (j_1, \dots, j_l) \mid w > = \Theta(a_{i_1}, \dots, a_{j_1}) \dots \Theta(a_{j_{l+1}}, \dots, a_{i_n}), \quad \forall j_1 + \dots + j_l = n \quad (26)$$

with the convention that $\Theta(a) = a, \forall a \in H_L$. Here the ... denotes the conatation of the several $\Theta(\cdot)$ so obtained.

Now we let $\text{exp} : H_W \rightarrow H_W$ be a linear map such that $\text{exp}(1) = 1$ and for any non empty word w ,

$$\text{exp}(w) = \sum_{(j_1, \dots, j_l) \in C(l(w))} \frac{1}{j_1! \dots j_l!} \langle j_1, \dots, j_l \mid w \rangle. \quad (27)$$

Proposition 7 *The map exp is an isomorphism between $(H_W, \sqcup, \emptyset, \Delta, \delta_\emptyset, S)$ and $(H_W, \sqcup_\Theta, \emptyset, \Delta, \delta_\emptyset, S_\Theta)$.*

This is shown in [2].

The primitives elements for both Hopf algebras (as they share the same coproduct) are the letters ([1]). We will spend the next lines to characterize some indecomposable elements.

Definition 11 $\forall u, v \in H_W$, we call Lie bracket the bracket $[u, v] \equiv uv - vu$.

Definition 12 *The elements generated by Lie brackets of letters are called Lie polynomials. A Lie polynomial is said to be of degree n if it is an iterated bracket of n different letters.*

In particular, the Lie polynomial of degree 1 are the letters, and if P is a Lie polynomial of degree n , $l(P) = n$. In the following P_n denotes a Lie polynomial of degree n .

Proposition 8 *The Lie polynomials can be seen as the indecomposable elements of $(H_W, \sqcup, \emptyset, \Delta, \delta_\emptyset, S)$.*

And this, instead of the Lyndon words. It will be the case in the rest of the article. We have the following corollary:

Corollary 2 *Let P be a Lie polynomial. Then $\text{exp}(P)$ can be seen as an indecomposable element of $(H_W, \sqcup_\Theta, \emptyset, \Delta, \delta_\emptyset, S_\Theta)$.*

This means exactly that any word in $(H_W, \sqcup_\Theta, \emptyset, \Delta, \delta_\emptyset, S_\Theta)$ can be written as a sum of products of the exp of Lie polynomials. Again, this is shown in [1] for the proposition and [2] for the corollary.

We ask the reader to focus his attention on a few things coming from the preceding propositions. First, the map exp respects the graduation defining by $|\cdot|$ as well as the coradical filtration and the filtration induced by the length of words. Secondly, if P is a Lie polynomial of degree n then $|P|$ is equal to the sum of the degrees (with respect to $|\cdot|$) of the letters composing it (e.g $P = [a_{i_1}, [\dots, [a_{i_{n-1}}, a_{i_n}]\dots]] \Rightarrow |P| = |a_{i_1}| + \dots + |a_{i_n}|$). Furthermore, we have that $\text{exp}(P) = P +$ some words of length strictly less than $l(P)$. Finally, for any word w , $\text{cor}(w) \leq l(w) \leq |w|$ with $\text{cor}(w)$ denoting the coradical degree of w .

These remarks allow us to decompose any word w in $(H_W, \sqcup_\Theta, \emptyset, \Delta, \delta_\emptyset, S_\Theta)$ as follows:

$$w = \sum_{(i_1, \dots, i_l) \in C(l(w))} \sum_{j_{i_1}, \dots, j_{i_l}} P_{j_{i_1}} \sqcup \dots \sqcup P_{j_{i_l}} + \text{words of length } < l(w). \quad (28)$$

Notice carefully that we use the shuffle product in the preceding equation and not the quasi shuffle product, as the Θ -part of the quasi shuffle product decreases the length of words at least by one.

Last but not least, we want to point out that for any letters a_{i_1}, \dots, a_{i_n} , $a_{i_1} \sqcup \dots \sqcup a_{i_n}$ gives rise to a sum of $n!$ words, each of these corresponding to a permutation of n elements. In other words, if we let S_n be the group of permutations of n elements and s a permutation of these n elements, one has:

$$a_{i_1} \sqcup \dots \sqcup a_{i_n} = \sum_{s \in S_n} a_{s(i_1)} \dots a_{s(i_n)}. \quad (29)$$

3.1 Feynman rules in the Hopf algebra of words

In this section (and in general in all the next sections) we denote by H_W the Hopf algebra $(H_W, \sqcup_\Theta, \emptyset, \Delta, \delta_\emptyset, S_\Theta)$; and we keep the notations of setion 1. In particular, we will denote \emptyset by $\mathbf{1}$, δ_\emptyset by $\hat{\mathbf{1}}$ and the reduced coproduct by $\tilde{\Delta}$.

Consider the set $\mathbb{C}[L]$ of polynomial in indeterminate L . An element $\phi \in G(H_W, \mathbb{C}[L])$ is called Feynman rules. It is generated by an element $\sigma \in g(H_W, \mathbb{C})$ by the formula

$$\phi = \exp_\star(L\sigma). \quad (30)$$

Notice that ϕ is entirely determined by the value of σ on the indecomposable elements of H_W , namely the images by exp of Lie polynomials. In the following, we suppose ϕ and σ fixed, and suppose that σ does not vanish on any element of H_L , so let say that $\forall a_i \in H_L, \sigma(a_i) = c_i \in \mathbb{C}$.

According to the Feynman rules, one can define a graduation on the Hopf algebra H_W , called kinematical graduation. An element $x \in H$ is called of kinematical degree n if $\deg(\phi(x)) = n$, where \deg denotes the usual degree of a polynomial in its indeterminate.

Proposition 9 *An element of coradical degree n is at most of kinematical degree n .*

Proof :

We have $\sigma(\mathbf{1}) = 0$, it follows:

$$\forall x \in H, \sigma^{\star n}(x) = m(\sigma \otimes \dots \otimes \sigma) \Delta^n(x) = m(\sigma \otimes \dots \otimes \sigma) \tilde{\Delta}^n(x).$$

□

Theorem 1 *Let $w = a_{i_1} \dots a_{i_n}$ be a connected word of length n . Then its kinematical degree is at most n and the coefficient of L^n in $\phi(w)$ is proportional to $\phi(a_{i_1} \sqcup_\Theta \dots \sqcup_\Theta a_{i_n}) = c_{i_1} \dots c_{i_n}$.*

Proof :

The first part of the theorem is trivial, as $cor(w) \leq l(w)$ and by applying proposition 9. For the second part, we decompose w as in equation (29). In this decomposition, the words of length strictly less than $l(w) = n$, as they also have a coradical degree strictly less than n , will not contribute to the coefficient of L^n . Furthermore, as the target algebra $\mathbb{C}[L]$ is commutative, each Lie polynomial of degree ≥ 2 will be mapped to 0 by σ^{*n} . The result follows. \square

3.2 Combinatorial Dyson Schwinger equations

Before defining what are the Combinatorial Dyson Schwinger equations in the Hopf algebra of words, we have to look at what are exactly the Hochschild-1-cocycles in the Hopf algebra $(H_W, \sqcup_\Theta, \mathbf{1}, \Delta, \hat{\mathbf{1}}, S_\Theta)$. Remember from 1.1 that we focus on operator $B_+^{a_r}$ such that $B_+^{a_r}(\mathbf{1}) = a_r$. Remember also that by definition, $\Delta(w) = \sum_{uv=w} v \otimes u$. So we have, for any non empty word w ,

$$\Delta B_+^{a_r}(w) = (id \otimes B_+^{a_r})\Delta(w) + B_+^{a_r}(w) \otimes \mathbf{1} \quad (31)$$

$$= \sum_{uv=w} v \otimes B_+^{a_r}(u) + B_+^{a_r}(w) \otimes \mathbf{1} \quad (32)$$

and, in the second hand,

$$\Delta B_+^{a_r}(w) = \sum_{u'v'=B_+^{a_r}(w)} v' \otimes u' \quad (33)$$

In particular, one gets that $w \otimes a_r$ must appear in $\Delta B_+^{a_r}(w)$, thus that $a_r w = B_+^{a_r}(w)$ and thus that $B_+^{a_r}$ is the operator which adds a_r at the beginning of each word.

Let $H_W[[\alpha]]$ be the ring of formal series with parameter α and coefficients in H_W . We define Combinatorial Dyson Schwinger equations as equations for these formal series living in $H_W[[\alpha]]$. As we consider a system of $R \in \mathbb{N}^*$ such equations, we define several series denoted by X^r , $r \in \{1, \dots, R\}$. The Combinatorial Dyson Schwinger equations of our interest are:

$$X^r = \mathbf{1}_W + sign(\eta_r)\alpha B_+^{a_r}(X^r Q), \quad (34)$$

$$Q = \prod_{r'=1}^R (X^{r'})^{\eta_{r'}}. \quad (35)$$

Thus, we extend the Feynman rules define in 2.1 to the $X \in H_W[[\alpha]]$:

$$\begin{aligned} \phi & : H_W[[\alpha]] \rightarrow \mathbb{C}[[L]][[\alpha]] \\ X & = \sum_n w_n \alpha^n \mapsto \sum_n \phi(w_n) \alpha^n. \end{aligned} \quad (36)$$

We call $G^r(\alpha, L) = \phi(X^r) = \sum_{i,j=0}^\infty b_{i,j}^r \alpha^i L^j$ with coefficients $b_{i,j} \in \mathbb{C}$. $G^r(\alpha, L)$ is called the log expansion of the series X^r .

Proposition 10 Consider $\{X^r = \sum_n w_n^r \alpha^n\}_{r \in R}$ a solution of the system of CDSE's (34). Then any of the w_n^r is of length n .

Proof:

We proceed by induction on the order in n of X^r . For $n = 0$, $w_0^r = 1_W$ for any r . Now consider that for a fixed n any of the $w_m^r, m \leq n$ is of length m . Now w_{n+1}^r is the coefficient of the term α^{n+1} in X^r . Thus it is equal to $\alpha B_+^{a_r}(W_n^r)$ where W_n^r is the term of order α^n in the product $X^r Q$. $X^r Q$ can be expanded in a formal serie in each of its variable, so we can write $W_n^r = \sum_i \prod_{j,r'} w_{i,j}^{r'}$ with $\sum j = n$, respecting the order in α . Hence W_n^r is of length n for each r . As $B_+^{a_r}$ increases the length by 1, w_{n+1}^r is of length $n+1$ and the proposition is proved. \square

Remark: By the same induction procedure, one can easily show that any of the $w_n^r, n \geq 1$ can be written as a word beginning by a_r .

The following is a direct consequence of the preceding proposition and theorem 1.

Corollary 3 Let $\{X^r = \sum_n w_n^r \alpha^n\}_{r \in R}$ a solution of the system of CDSE's (34). Then any $G^r(\alpha, L)$ can be written as $\sum_{i=0}^{\infty} \sum_{j=0}^i b_{i,j}^r \alpha^i L^j$.

We call $G_{LL}^r(\alpha, L)$ the sum $\sum_{i=0}^{\infty} b_{i,i}^r \alpha^i L^i$. $G_{LL}^r(\alpha, L)$ is called the leading log expansion of the series X^r .

Proposition 11 Let $w = a_{i_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} a_{i_n}, a_{i_1}, \dots, a_{i_n} \in H_L$. Then the coefficient of the term in L^{n+1} in $\phi(B_+^{a_r}(w))$ is equal to $\frac{1}{n+1} \phi(a_{i_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} a_{i_n})$.

Proof:

Denoting as above by S_n the group of permutations of n elements, we know that $a_{i_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} a_{i_n} = \sum_{s \in S_n} a_{s(i_1)} \dots a_{s(i_n)} + \text{words of length at most } n-1$. By applying $B_+^{a_r}$ on these words, we obtain $n!$ words beginning with a_r of length $n+1$.

Now consider the product $a_r \sqcup_{\Theta} a_{i_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} a_{i_n}$. You obtain $n!$ words beginning with a_r , $n * n!$ other words of length $n+1$ and some other words of length at most n , which will be mapped to polynomial of degree at most n by ϕ . For the other words of length $n+1$, bring a_r to the first place by applying the transformation $a_i a_r = a_r a_i + [a_i, a_r]$. You then obtain $(n+1)!$ words beginning with a_r , and $n * n!$ words containing Lie polynomials of degree $n+1$ which will be mapped again to a polynomial of degree n in L by ϕ thanks to the same argument as in the proof of Theorem 1. In fact, the $(n+1)!$ words beginning with a_r are exactly $n+1$ times the words we obtained by computing $B_+^{a_r}(w)$. Applying Theorem 1 on both sides ends to prove the result. \square

Theorem 2 Consider $\{X^r = \sum_n w_n^r \alpha^n\}_{r \in R}$ a solution of the system of CDSE's (34). Then the leading log expansion of each X^r obeys

$$\sum_{q_1, \dots, q_R; q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r \phi(a_1)^{q_1} \dots \phi(a_r)^{q_r-1} \dots \phi(a_R)^{q_R} \alpha^{q_1 + \dots + q_R} \\ = G_{LL}^r \prod_{r'=1}^R (G_{LL}^{r'})^{\eta_{r'}} \quad (37)$$

with some coefficients $C_{q_1, \dots, q_R}^r \in \mathbb{C}$.

This is proved in a more general case in the appendix.

Going on with this relation, one gets :

$$\sum_{q_1, \dots, q_R; q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r (c_1 L \alpha)^{q_1} \dots (c_r L \alpha)^{q_r-1} \dots (c_R L \alpha)^{q_R} \\ = G_{LL}^r \prod_{r'=1}^R (G_{LL}^{r'})^{\eta_{r'}}. \quad (38)$$

The left-hand-side can be written as

$$\sum_{q_1, \dots, q_R; q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r (c_1 L \alpha)^{q_1} \dots (c_r L \alpha)^{q_r-1} \dots (c_R L \alpha)^{q_R} \quad (39) \\ = \sum_{q_1, \dots, q_R; q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r c_1^{q_1} \dots c_r^{q_r-1} \dots c_R^{q_R} (L \alpha)^{q_1 + \dots + q_R-1} \quad (40) \\ = \frac{1}{c_r} \sum_{q_1, \dots, q_R; q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r c_1^{q_1} \dots c_r^{q_r} \dots c_R^{q_R} (L \alpha)^{q_1 + \dots + q_R-1} \quad (41) \\ = \frac{1}{c_r} \frac{\partial}{\partial L \alpha} G_{LL}^r(L, \alpha). \quad (42)$$

Hence, we have to solve the following system of differential equations:

$$\frac{\partial}{\partial L \alpha} G_{LL}^r(L, \alpha) = c_r G_{LL}^r \prod_{r'=1}^R (G_{LL}^{r'})^{\eta_{r'}}. \quad (43)$$

This is quite trivial and is achieved defining $x = \alpha L$ and $y_r(x) = \ln G_{LL}^r(\alpha, L)$.

The boundary conditions are $y^r(0) = 0$. Then we can write

$$y'_r = c_r \exp \left(\sum_{r'=1}^R \eta_{r'} y_{r'} \right) \quad (44)$$

$$y'_{r_1} = \frac{c_{r_1}}{c_{r_2}} y'_{r_2} \quad (45)$$

$$y_{r_1} = \frac{c_{r_1}}{c_{r_2}} y_{r_2} \quad (46)$$

$$y'_r = c_r \exp \left(\sum_{r'=1}^R \eta_{r'} \frac{c_{r'}}{c_r} y_r \right) \quad (47)$$

$$A = \sum_{r'=1}^R \eta_{r'} c_{r'} \quad (48)$$

$$y'_r = c_r \exp \left(\frac{A}{c_r} y_r \right) \quad (49)$$

We solve the last differential equation for y , and finally take the exponential of it to find

$$\begin{aligned} G_{LL}^r &= (1 + A\alpha L)^{-\frac{c_r}{A}} \text{ if } \eta_r < 0 \\ G_{LL}^r &= (1 - A\alpha L)^{\frac{c_r}{A}} \text{ if } \eta_r > 0 \end{aligned}$$

4 Outlook

We gave here an example on how one can treat a system of combinatorial Dyson Schwinger equations, using techniques of [7]. This allows to compute the leading log expansion of such a system with a nice system of order 1 differential equations. Further work about it will to consider cases with two scales Green's functions (case of general three points Green's functions) or combining several interactions together, which will also include to consider several coupling constants. The equations one will obtain (if one will !) will be partial differential equations, much more complicated to solves, but this could teach more on the combinatoric structures and approximations of Green's functions.

Appendix

Here, we compute a general system of combinatorial Dyson Schwinger equation of the form

$$X^r = \mathbb{1}_H + \alpha B_+^{\gamma^r} [f^r(X^1, \dots, X^R)], \quad (50)$$

with f^r a function which can be expanded in a formal series in each of its variable. We proves with a (quite tedious) direct derivation that the leading log

expansion of the solution of this system is a solution of the following system:

$$\sum_{q_1, \dots, q_R; q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r \phi_R(a_1)^{q_1} \dots \phi_R(a_r)^{q_r-1} \dots \phi_R(a_R)^{q_R} \alpha^{q_1 + \dots + q_R} = f^r(G_{LL}^1, \dots, G_{LL}^R). \quad (51)$$

About the notation, we write $\delta(m - n)$ for the Kronecker delta $\delta_{m,n}$. We also asserts that Proposition 10, Corollary 3 and Proposition 11 are still truth in this context; in fact, the preceding proofs can be easily extended to this case.

$$\begin{aligned} X^r &= \mathbb{1} + \alpha B_+^{a_r} [f^r(X^1, \dots, X^R)]. \\ \sum_m w_m^r \alpha^m &= \mathbb{1} + \alpha B_+^{a_r} \left[\sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r X^{1 \sqcup_{\Theta} n_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} X^{R \sqcup_{\Theta} n_R} \right] \\ \sum_m w_m^r \alpha^m &= \mathbb{1} + \alpha B_+^{a_r} \left[\sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \left(\sum_{m_1} w_{m_1}^{r_1} \alpha^{m_1} \right)^{\sqcup_{\Theta} n_1} \sqcup_{\Theta} \dots \right. \\ &\quad \left. \sqcup_{\Theta} \left(\sum_{m_R} w_{m_R}^{r_R} \alpha^{m_R} \right)^{\sqcup_{\Theta} n_R} \right]. \\ \sum_m w_m^r \alpha^m &= \mathbb{1} + \alpha B_+^{a_r} \left[\sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \left(\sum_{m_{1,1}} w_{m_{1,1}}^{r_1} \alpha^{m_{1,1}} \right) \sqcup_{\Theta} \dots \right. \\ &\quad \sqcup_{\Theta} \left(\sum_{m_{1,n_1}} w_{m_{1,n_1}}^{r_1} \alpha^{m_{1,n_1}} \right) \sqcup_{\Theta} \dots \sqcup_{\Theta} \left(\sum_{m_{R,1}} w_{m_{R,1}}^{r_R} \alpha^{m_{R,1}} \right) \sqcup_{\Theta} \dots \\ &\quad \left. \sqcup_{\Theta} \left(\sum_{m_{R,n_R}} w_{m_{R,n_R}}^{r_R} \alpha^{m_{R,n_R}} \right) \right]. \end{aligned}$$

Now we can use the linearity of the B_+ operator to write:

$$\begin{aligned} \sum_m w_m^r \alpha^m &= \mathbb{1} + \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \sum_{m_{1,1}} \dots \\ &\quad \sum_{m_{R,n_R}} \alpha^{1+m_{1,1}+\dots+m_{R,n_R}} B_+^{a_r} \left[w_{m_{1,n_1}}^{r_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} w_{m_{R,n_R}}^{r_R} \right]. \end{aligned}$$

This allows us to take the equality order by order

$$\begin{aligned} w_0^r &= \mathbb{1}, \\ w_m^r &= \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R} \sum_{m_{1,1}} \dots \sum_{m_{R,n_R}} \delta(1 + m_{1,1} + \dots + m_{R,n_R} - m) \\ &\quad B_+^{a_r} \left[w_{m_{1,1}}^{r_1} \sqcup_{\Theta} \dots \sqcup_{\Theta} w_{m_{R,n_R}}^{r_R} \right]. \end{aligned}$$

From Theorem 1, one knows that one can consider only words written as $a_1^{\sqcup \ominus q_1} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R}$ in w_m^r :

$$w_m^r = \sum_{q_1, \dots, q_R} C_{q_1, \dots, q_R}^r a_1^{\sqcup \ominus q_1} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R} \delta(q_1 + \dots + q_R - m) + O(m-1),$$

where $O(m-1)$ stands for words of kinematical degree at most $m-1$. One can then rewrite, keeping only the words of kinematical degree m ,

$$\begin{aligned} & \sum_{q_1, \dots, q_R} C_{q_1, \dots, q_R}^r a_1^{\sqcup \ominus q_1} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R} \delta(q_1 + \dots + q_R - m) = \\ & \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \sum_{m_{1,1}} \dots \sum_{m_{R, n_R}} \delta(1 + m_{1,1} + \dots + m_{R, n_R} - m) \\ & B_+^{a_r} \left[\sum_{q_1, m_{1,1}, \dots, q_R, m_{1,1}} C_{q_1, m_{1,1}, \dots, q_R, m_{1,1}}^{r_1} a_1^{\sqcup \ominus q_1, m_{1,1}} \delta(q_1, m_{1,1} + \dots + q_R, m_{1,1} - m_{1,1}) \sqcup \ominus \dots \right. \\ & \left. \sqcup \ominus a_R^{\sqcup \ominus q_R, m_{1,1}} \sqcup \ominus \dots \sqcup \ominus \sum_{q_1, m_{R, n_R}, \dots, q_R, m_{R, n_R}} C_{q_1, m_{R, n_R}, \dots, q_R, m_{R, n_R}}^{r_R} \right. \\ & \left. a_1^{\sqcup \ominus q_1, m_{R, n_R}} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R, m_{R, n_R}} \delta(q_1, m_{R, n_R} + \dots + q_R, m_{R, n_R} - m_{R, n_R}) \right]. \end{aligned}$$

One more time one uses the linearity of the B_+ operator to write

$$\begin{aligned} & \sum_{q_1, \dots, q_R} C_{q_1, \dots, q_R}^r a_1^{\sqcup \ominus q_1} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R} \delta(q_1 + \dots + q_R - m) = \\ & \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \sum_{m_{1,1}} \dots \sum_{m_{R, n_R}} \delta(1 + m_{1,1} + \dots + m_{R, n_R} - m) \\ & \sum_{q_1, m_{1,1}, \dots, q_R, m_{1,1}} C_{q_1, m_{1,1}, \dots, q_R, m_{1,1}}^{r_1} \delta(q_1, m_{1,1} + \dots + q_R, m_{1,1} - m_{1,1}) \dots \\ & \sum_{q_1, m_{R, n_R}, \dots, q_R, m_{R, n_R}} C_{q_1, m_{R, n_R}, \dots, q_R, m_{R, n_R}}^{r_R} \delta(q_1, m_{R, n_R} + \dots + q_R, m_{R, n_R} - m_{R, n_R}) \\ & B_+^{a_r} [a_1^{\sqcup \ominus q_1, m_{1,1} + \dots + q_1, m_{R, n_R}} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R, m_{1,1} + \dots + q_R, m_{R, n_R}}]. \end{aligned}$$

Use Proposition 11 to rewrite:

$$\begin{aligned}
& (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r a_1^{\sqcup \ominus q_1} \sqcup \ominus \dots \sqcup \ominus a_R^{\sqcup \ominus q_R} \delta(q_1 + \dots + q_R - m) = \\
& \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \sum_{m_{1,1}} \dots \sum_{m_{R, n_R}} \delta(1 + m_{1,1} + \dots + m_{R, n_R} - m) \\
& \sum_{q_{1, m_{1,1}}, \dots, q_{R, m_{1,1}}} C_{q_{1, m_{1,1}}, \dots, q_{R, m_{1,1}}}^{r_1} \delta(q_{1, m_{1,1}} + \dots + q_{R, m_{1,1}} - m_{1,1}) \dots \\
& \sum_{q_{1, m_{R, n_R}}, \dots, q_{R, m_{R, n_R}}} C_{q_{1, m_{R, n_R}}, \dots, q_{R, m_{R, n_R}}}^{r_R} \delta(q_{1, m_{R, n_R}} + \dots + q_{R, m_{R, n_R}} - m_{R, n_R}) \\
& a_1^{\sqcup \ominus q_{1, m_{1,1}} + \dots + q_{1, m_{R, n_R}}} \sqcup \ominus \dots \sqcup \ominus a_r^{\sqcup \ominus q_{r, m_{1,1}} + \dots + q_{r, m_{R, n_R}} + 1} \sqcup \ominus \dots \\
& \sqcup \ominus a_R^{\sqcup \ominus q_{R, m_{1,1}} + \dots + q_{R, m_{R, n_R}}} \delta(q_{1, m_{1,1}} + \dots + q_{1, m_{R, n_R}} - q_1) \dots \\
& \delta(q_{r, m_{1,1}} + \dots + q_{r, m_{R, n_R}} + 1 - q_r) \dots \delta(q_{R, m_{1,1}} + \dots + q_{R, m_{R, n_R}} - q_R).
\end{aligned}$$

The next step is to apply the Feynman rules. Moreover, we reorganize the sum in order to put each coefficient C in front of its corresponding term in the sum.

$$\begin{aligned}
& (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r \phi_R(a_1)^{q_1} \dots \phi_R(a_R)^{q_R} \delta(q_1 + \dots + q_R - m) = \\
& \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \sum_{m_{1,1}} \dots \sum_{m_{R, n_R}} \delta(1 + m_{1,1} + \dots + m_{R, n_R} - m) \\
& \sum_{q_{1, m_{1,1}}, \dots, q_{R, m_{1,1}}} C_{q_{1, m_{1,1}}, \dots, q_{R, m_{1,1}}}^{r_1} \phi_R(a_1)^{q_{1, m_{1,1}}} \dots \phi_R(a_R)^{q_{R, m_{1,1}}} \\
& \delta(q_{1, m_{1,1}} + \dots + q_{R, m_{1,1}} - m_{1,1}) \dots \\
& \sum_{q_{1, m_{R, n_R}}, \dots, q_{R, m_{R, n_R}}} C_{q_{1, m_{R, n_R}}, \dots, q_{R, m_{R, n_R}}}^{r_R} \phi_R(a_1)^{q_{1, m_{R, n_R}}} \dots \phi_R(a_R)^{q_{R, m_{R, n_R}}} \\
& \delta(q_{1, m_{R, n_R}} + \dots + q_{R, m_{R, n_R}} - m_{R, n_R}) \phi_R(a_r) \delta(q_{1, m_{1,1}} + \dots + q_{1, m_{R, n_R}} - q_1) \dots \\
& \delta(q_{r, m_{1,1}} + \dots + q_{r, m_{R, n_R}} + 1 - q_r) \dots \delta(q_{R, m_{1,1}} + \dots + q_{R, m_{R, n_R}} - q_R).
\end{aligned}$$

Then one divide by $\phi_R(a_r)$ both sides, which is allowed for $n \geq 1$ (in the trivial case $n = 0$, $\phi_R(w_0^r) = 1$):

$$\begin{aligned}
& (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r \phi_R(a_1)^{q_1} \dots \phi_R(a_r)^{q_r-1} \dots \phi_R(a_R)^{q_R} \delta(q_1 + \dots + q_R - m) = \\
& \sum_{n_1, \dots, n_R} f_{n_1, \dots, n_R}^r \sum_{m_{1,1}} \dots \sum_{m_{R, n_R}} \delta(1 + m_{1,1} + \dots + m_{R, n_R} - m) \\
& \sum_{q_{1, m_{1,1}}, \dots, q_{R, m_{1,1}}} C_{q_{1, m_{1,1}}, \dots, q_{R, m_{1,1}}}^{r_1} \phi_R(a_1)^{q_{1, m_{1,1}}} \dots \phi_R(a_R)^{q_{R, m_{1,1}}} \\
& \delta(q_{1, m_{1,1}} + \dots + q_{R, m_{1,1}} - m_{1,1}) \dots \\
& \sum_{q_{1, m_{R, n_R}}, \dots, q_{R, m_{R, n_R}}} C_{q_{1, m_{R, n_R}}, \dots, q_{R, m_{R, n_R}}}^{r_R} \phi_R(a_1)^{q_{1, m_{R, n_R}}} \dots \phi_R(a_R)^{q_{R, m_{R, n_R}}} \\
& \delta(q_{1, m_{R, n_R}} + \dots + q_{R, m_{R, n_R}} - m_{R, n_R}) \\
& \delta(q_{1, m_{1,1}} + \dots + q_{1, m_{R, n_R}} - q_1) \dots \delta(q_{r, m_{1,1}} + \dots + q_{r, m_{R, n_R}} + 1 - q_r) \dots \\
& \delta(q_{R, m_{1,1}} + \dots + q_{R, m_{R, n_R}} - q_R).
\end{aligned}$$

And we do the whole resummation over q_1, \dots, q_R , re-injecting also the α ; the problematic case $q_r = 0$ being cancelled by $\delta(q_{r, m_{1,1}} + \dots + q_{r, m_{R, n_R}} + 1 - q_r)$. Hence,

$$\begin{aligned}
& \sum_{q_1, \dots, q_R: q_r \geq 1} (q_1 + \dots + q_R) C_{q_1, \dots, q_R}^r \phi_R(a_1)^{q_1} \dots \phi_R(a_r)^{q_r-1} \dots \phi_R(a_R)^{q_R} \alpha^{q_1 + \dots + q_R} \\
& = f^r(G_{LL}^1, \dots, G_{LL}^R).
\end{aligned}$$

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